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Contour Dynamics for the Euler Equations in Two Dimensions

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We present a contour dynamics algorithm for the Euler equations of fluid dynamics in two dimensions. This is applied to regions of piecewise-constant vorticity within *finite-area-vortex regions* (FAVR's). Essentially, this reduces the dimensionality by one and we are treating the interaction of closed polygonal contours whose nodes are advected by the total fluid motion computed self-consistently. A leapfrog centered scheme is used for temporal advancement. Computer simulation results are given for two and four like-signed interacting FAVR's. In all cases wavelike surface deformations are observed. If the distance between FAVR's is comparable to their extent ("diameter"), these surface deformations are large. They play an essential role in the observed coalescence of FAVR's.

1. INTRODUCTION

High Reynold's number flows in two dimensions almost always develop *finite-area-vortex-regions* (FAVR's) with steep sides. The evolution of these incompressible flows involves the self- and mutual interaction of these *deformable* FAVR's.

When finite-difference methods are used to simulate these flows, a high mesh resolution is required to avoid introducing grid-scale dissipation and dispersion errors. To overcome these difficulties, Chorin [1] has proposed a vortex scheme which advects *nondeformable* FAVR's with a velocity composed of two parts: a deterministic component calculated from the existing vorticity distribution; and a zero-mean Gaussian random component to simulate dissipation. The advantages and disadvantages of the proposed method are under active investigation for shear flows [2] and cavity flows [3].

The motivation for the present study arose when Christiansen and Zabusky [4] studied the stability of the inviscid von Kármán wake in two dimensions. They used

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a two-dimensional point vortex-field code and noted the important effects of induced waves on the surface of FAVR's. To elucidate these nonlinear wave effects with a moderate amount of computation we introduce the method of contour dynamics (CD) for inviscid incompressible fluids in two dimensions.

The CD method does not use an underlying lattice and is a generalization of the "water-bag" model used to study plasma dynamics [5, 6]. In essence, it amounts to a dynamic interaction among closed contours enclosing FAVR's. That is, we have reduced the dimensionality by one. To obtain this great simplification, we assume that each FAVR has a constant vorticity density of arbitrary magnitude.

In this paper we present computer simulation results for one, two, and four interacting like-signed FAVR's. The latter two simulations show large-amplitude wavelike deformations resulting from self- and mutual interactions. We believe these deformations play an essential role in the stability and coalescence phenomena observed for coherent structures in fluids [7]. Generalizations of the method and error analyses will be described in future publications.

2. Contour Dynamics Algorithm

A. *Continuum Formulation*

The incompressible, inviscid Navier–Stokes equations (Euler equations) in two dimensions can be written in "vorticity" form as the coupled set of equations

$$\omega_t + u\omega_x + v\omega_y = 0, \tag{1}$$

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = -\omega, \tag{2}$$

and

$$\begin{gathered} u = \psi_y, \qquad v = -\psi_x, \\ \omega = -u_y + v_x. \end{gathered} \tag{3}$$

Regions of positive vorticity or circulation correspond to counterclockwise motions of convected fluid elements; that is, we are studying a right-handed coordinate system where the vorticity vector is along $\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$.

We write the stream function at (x, y) as an integral over all the FAVR's

$$\psi(x, y) = -(1/2\pi) \int da\, \omega(\xi, \eta) \log(r/R), \tag{4}$$

where $da = d\xi\, d\eta$,

$$r^2 = (x - \xi)^2 + (y - \eta)^2, \tag{5}$$

and $\log(r/R)$ is the Green's function of Poisson's equation (2). We have inserted the normalizing constant R for convenience in writing algorithms. All pertinent quantities are depicted in Fig. 1. The velocity at any point in the flow field, in particular on closed contours, is

$$\mathbf{u} = \nabla \times \mathbf{e}_z\psi = \mathbf{e}_x\psi_y - \mathbf{e}_y\psi_x\,, \tag{6}$$

or

$$\mathbf{u} = (2\pi)^{-1} \int da\ \omega[\mathbf{e}_x\partial_\eta \log(r/R) - \mathbf{e}_y\partial_\xi \log(r/R)], \tag{7}$$

$$\partial_y \to -\partial_\eta \qquad \text{and} \qquad \partial_x \to -\partial_\xi$$

because of the definition (5). Integrating (7) by parts yields

$$\mathbf{u} = -(2\pi)^{-1} \int da \log(r/R)[\mathbf{e}_x\ \partial_\eta\omega - \mathbf{e}_y\ \partial_\xi\omega], \tag{8}$$

$$= (2\pi)^{-1}\ \mathbf{e}_z \times \int da(\log(r/R))\ \nabla_\xi\omega. \tag{9}$$

For clarity, we append at times the subscript ξ or x to the gradient and Laplacian operators. That is,

$$\nabla_\xi = \mathbf{e}_x\partial_\xi + \mathbf{e}_y\partial_\eta \qquad \text{and} \qquad \nabla_x = \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y\,.$$

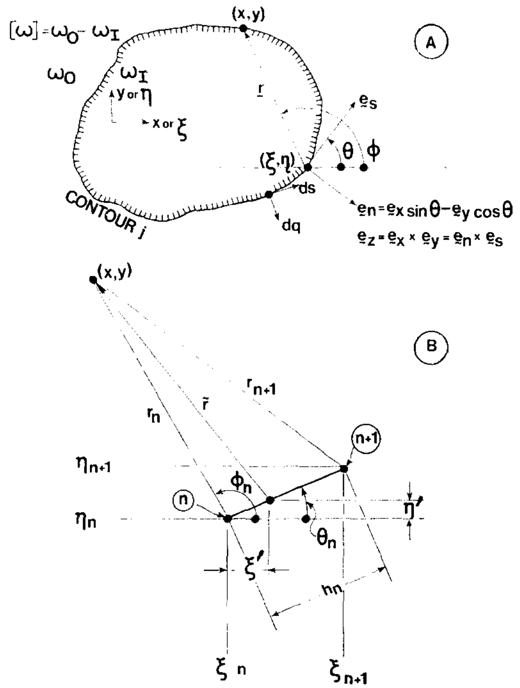

FIG. 1. (A) Definition of quantities for a finite-area-vortex-region (FAVR) of strength ω_I. (B) Definition of quantities for a piecewise-linear contour.

We decompose the integral of (8) into two parts: the large area within the contour where $\nabla_\xi \omega = 0$ contributes nothing; and the small strip following the contour where $\nabla_\xi \omega$ is singular yields a finite contribution. On the contour, we introduce the localized s, q orthogonal coordinate system as shown in Fig. 1, where area $dq\, ds = d\xi\, d\eta$. This transformation with unit Jacobian holds if the contour is piecewise continuous. (That is, the tangent and normal directions may jump discontinuously.) Thus,

$$\partial_\xi \omega = (\partial_q \omega)(dq/d\xi) \qquad \text{and} \qquad \partial_\eta \omega = (\partial_q \omega)(dq/d\eta), \tag{10a}$$

or

$$\begin{aligned} \partial_\xi \omega &= [\omega]\, \delta(q) \cos(n, \xi) = [\omega]\, \delta(q) \sin \theta, \\ \partial_\eta \omega &= [\omega]\, \delta(q) \cos(n, \eta) = -[\omega]\, \delta(q) \cos \theta, \end{aligned} \tag{10b}$$

where

$$[\omega] = \omega\,|_0 - \omega\,|_I, \qquad \text{(see Fig. 1 A)},$$

and $\delta(q)$ is the delta function with q measured positively in the $\mathbf{e}_n$ direction. The x and y components of the direction cosines are $\cos(n, \xi) = \sin \theta$ and $\cos(n, \eta) = -\cos \theta$. If we replace the components of $\nabla_\xi \omega$ using (10b) and perform the q integration, we obtain

$$\mathbf{u} = (2\pi)^{-1} \sum_{j=1}^{N_c} [\omega]_j \oint_j \log \frac{r}{R} [\mathbf{e}_x \cos \theta_j + \mathbf{e}_y \sin \theta_j]\, ds_j,$$

$$\mathbf{u} = (2\pi)^{-1} \sum_{j=1}^{N_c} [\omega]_j \oint_j \log \frac{r}{R} [\mathbf{e}_x\, d\xi_j + \mathbf{e}_y\, d\eta_j], \tag{11}$$

where $[\omega]_j$ is the value of $[\omega]$ associated with contour j. Hence the constant vorticity regions have been replaced by a distribution of sources with logarithmic strengths along contours (labeled with the index j) surrounding N_c regions in the field.

2B. *Spatial Discretization*

We assume that the contour bounding one FAVR is a polygon with N nodes and now consider their interactions. As depicted in Fig. 1B, along each segment of length h_n between nodes n and $n+1$, r is approximated by $\tilde{r}$

$$\tilde{r}^2 = (x - \xi_n - \xi')^2 + (y - \eta_n - \xi' \tan \theta_n)^2, \tag{12}$$

or

$$\tilde{r}^2 = r_n^2(1 - 2\zeta\mu_n + \zeta^2), \tag{13}$$

where r_n is the distance from node n to (x, y) (usually another node), ϕ_n is the associated angle and

$$\mu_n = \cos(\phi_n - \theta_n), \tag{14}$$

$$\zeta = (\xi - \xi_n)/(r_n \cos \theta_n) = \xi'/(r_n \cos \theta_n), \tag{15}$$

and

$$\tan \theta_n = (y_{n+1} - y_n)/(x_{n+1} - x_n).$$

For one FAVR, Eq. (11) can be discretized as

$$\mathbf{u} = \sum_{n=1}^{N} (\Delta u)_n \, (\mathbf{e}_x \cos \theta_n + \mathbf{e}_y \sin \theta_n) \tag{16}$$

(since θ_n and ϕ_n are identified with node n) and

$$(\Delta u)_n = (1/4\pi)[\omega] \, r_n \int_0^{(h_n/r_n)} \log(\tilde{r}/R)^2 \, d\zeta \tag{17}$$

is the increment contributed by the linear segment between nodes n and $n + 1$. After some algebra we write

$$(\Delta u)_n = (1/4\pi)[\omega] \, h_n\{\log(r_n/R)^2 + (1 - \mu_n h_n^{-1} r_n) \log q_n - 2 + 2(1 - \mu_n^2)^{1/2} (r_n/h_n) \tan^{-1} (\beta_n)\}, \quad (n \neq 1, N) \tag{18}$$

where

$$q_n = 1 - 2(h_n/r_n) \, \mu_n + (h_n/r_n)^2,$$

and

$$\beta_n = (h_n/r_n)(1 - \mu_n^2)^{1/2}/[1 - (h_n/r_n) \, \mu_n]. \tag{19}$$

Note we have used

$$\mathrm{Re}[\zeta_n \log(1 - h_n \zeta_n^*/r_n)] = \mu_n \log q_n^{1/2} - (1 - \mu_n^2)^{1/2} \tan^{-1} \beta_n \, ,$$

where

$$\zeta_n \text{ and } \zeta_n{}^* \text{ are roots of } \quad 0 = 1 - 2 \, \zeta \mu_n + \zeta^2.$$

If $n = 1$ is identified with the point (x, y) then

$$(\Delta u)_N = (2\pi)^{-1}[\omega] \, h_N(\log(h_N/R) - 1), \tag{20}$$

and

$$(\Delta u)_1 = (2\pi)^{-1}[\omega] \, h_1(\log(h_1/R) - 1). \tag{21}$$

Furthermore, the contribution from distant nodes ($h_n/r_n \ll 1$) may be obtained from an asymptotic expansion of (18) or more directly by expanding (17) in an asymptotic series and integrating,

$$(\Delta u)_n = ([\omega]/2\pi) \, h_n\{\log(r_n/R) - \tfrac{1}{2}\mu_n(h_n/r_n) + \tfrac{1}{6}(1 - 2\mu_n^2)(h_n/r_n)^2 + \tfrac{1}{4}\mu_n(1 - \tfrac{4}{3}\mu_n^2)(h_n/r_n)^3 + O(h_n/r_n)^4\}. \tag{22}$$

A similar procedure may be followed to find interactions among nodes on different contours.

2C. *Temporal Discretization*

We now identify (x, y) with a node of a FAVR, or (x_m, y_m). Each node is convected with the local fluid velocity, or

$$\mathbf{u}(x_m, y_m) = \dot{\mathbf{x}}_m = \dot{x}_m \mathbf{e}_x + \dot{y}_m \mathbf{e}_y$$
$$= \sum_{n=1}^{N} (\Delta u)_{nm} (\cos\theta_n \mathbf{e}_x + \sin\theta_n \mathbf{e}_y), \tag{23}$$

where $(\Delta u)_{nm}$ is $(\Delta u)_n$ of Eq. (18) with r_n replaced by

$$r_{nm} = [(x_m - x_n)^2 + (y_m - y_n)^2]^{1/2}, \tag{24a}$$

and ϕ_n is

$$\phi_{nm} = \tan^{-1}[(y_m - y_n)/(x_m - x_n)]. \tag{24b}$$

Since we are dealing with a conservative system, we integrate (23) using a centered leapfrog scheme, or

$$(2\Delta t)^{-1} (\mathbf{x}_m^{p+1} - \mathbf{x}_m^{p-1}) = \sum_{n=1}^{N} (\Delta u)_{nm}^{p} (\cos\theta_n{}^p \mathbf{e}_x + \sin\theta_n{}^p \mathbf{e}_y). \tag{25}$$

To start, one takes a forward time step on a reduced time interval, $(\Delta t)_s \ll \Delta t$, and then uses a doubling-up procedure. Δt_s is chosen so that the errors in the initial step are consistent with those in the later steps.

2D. *Outline of the Calculation*

Initially, the number of nodes N_j, shape and vorticity of each FAVR $(j = 1, \ldots J)$ is given. N_j is chosen sufficiently large to approximate a continuum curve.

Using (16), and summing in addition over the J FAVR's we obtain the velocity of each node. For convenience we have adopted two formulas for $(\Delta u)_{nm}$:

$$(1)\quad (\Delta u)_{nm} = (2\pi)^{-1}[\omega]\, h_m[\log(h_m/R) - 1], \qquad m = n - 1 \text{ or } n, \tag{26a}$$
$$(2)\quad (\Delta u)_{nm} = (2\pi)^{-1}[\omega]\, h_m[\log(r_m/R)], \qquad m \neq n - 1 \text{ or } n. \tag{26b}$$

The first is obtained from (20) and (21). The second comes from truncating the asymptotic series (22) after the first term. The errors introduced by this convenient truncation will be investigated at another time.

The new position of each node is obtained by solving (25), as indicated. Δt is chosen such that

$$\Delta t < \mathrm{Min}\{h_n/\mathrm{Max}\mid \mathbf{u}(x_n, y_n)\mid\}, \tag{27}$$

where the Min–Max is over all nodes. For example, in the runs described in Section 3 $\Delta t = 0.2$. For the duration of the runs performed the odd and even steps of the leapfrog solution did not fall out of "phase" and no temporal smoothing was used.

3. Computational Results

To validate the CD algorithm, we examined the rotation of a 2:1 ellipse with $N = 30$ and $\Delta t = 0.2$. As Lamb [8] reports, the period of rotation is $T = (2\pi/[\omega]) \cdot \{(b + a)^2/ba\}$ or 9π for a 2:1 ellipse with $[\omega] = 1.0$. The duration of the run was $t_F = 99.799$ and the observed rotation in Fig. 2c was 1272.3° (a little over $3\frac{1}{2}$ revolutions). Hence, the observed rate of rotation was 12.75°/unit-time compared to the true rate of $(360/9\pi) = 12.73$°/unit-time. One notices a slight difference in the shape of the ellipse, but overall the agreement is good.

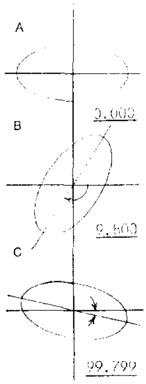

FIG. 2. The rotation of a Kirchoff elliptic vortex. Ratio of major-to-minor axis = 2:1. $N = 60$, $\Delta t = 0.2$.

Figure 3 shows four cases of the interaction of two identical negatively signed ($\omega_I = -1, \omega_0 = 0$) initially circular FAVR's of identical diameter $D = 0.6$. The initial separations of centers were $L_c = 1.10, 1.022, .02$, and 0.80, respectively, or $(D/L_c) = 0.5455, 0.5871, 0.5882$, and 0.75. Each circle was discretized with $N = 30$. At the largest separation, case 1, the FAVR's rotated about one another in a clockwise direction and induced wavelike perturbations on their surface. As the initial separation, L_c, is decreased, the deformations of the vortex surface increase. In case 2 they "pulsate" toward one another and then withdraw, leaving an extremely narrow region of vorticity between them.

In case 3 they coalesce weakly and exchange "vortex fluid." We see a narrow region of vorticity creeping around each FAVR. In case 4 they coalesce strongly and eject vortex "arms," a common feature of high Reynolds number fluid dynamic simulations. We stopped this run at this time because the spacing of nodes had increased beyond a tolerable amount.

In Fig. 4, we show the interaction of four identical negatively signed ($\omega_I = -1$, $\omega_0 = 0$) initially circular FAVR's of diameter $D = 1.0$ and $N = 20$ whose centers are placed on a circle of radius 1.16. This is a well-unknown unstable configuration and the manner of coalescence is exhibited.

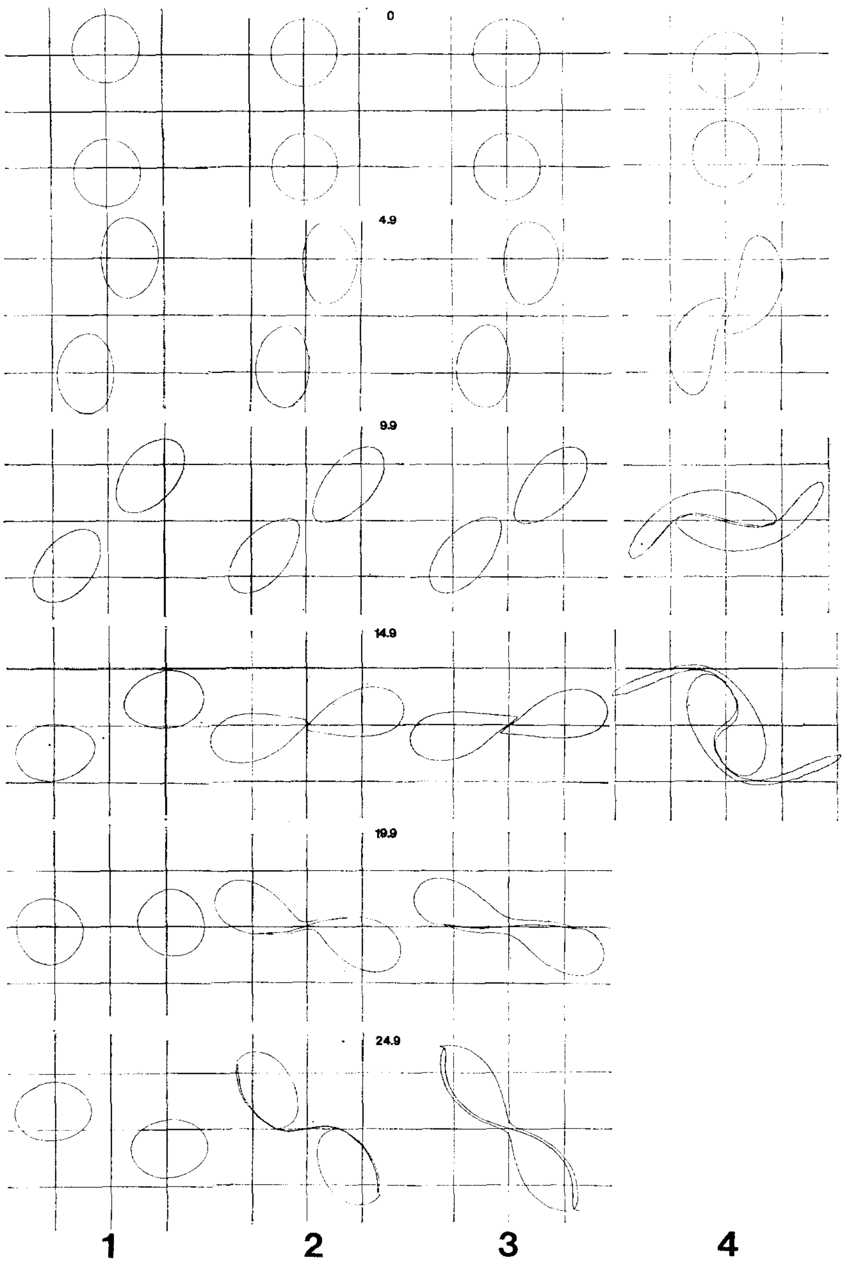

FIG. 3. Interaction of $J = 2$ initially circular negatively signed FAVR's of diameter $D = 0.6$. ($N = 30$ segments per circle, $\Delta t = 0.2$). Initial separations are: (1) 1.10; (2) 1.022; (3) 1.02; and (4) 0.80. $[\omega] = +1$ and the gross rotation is clockwise.

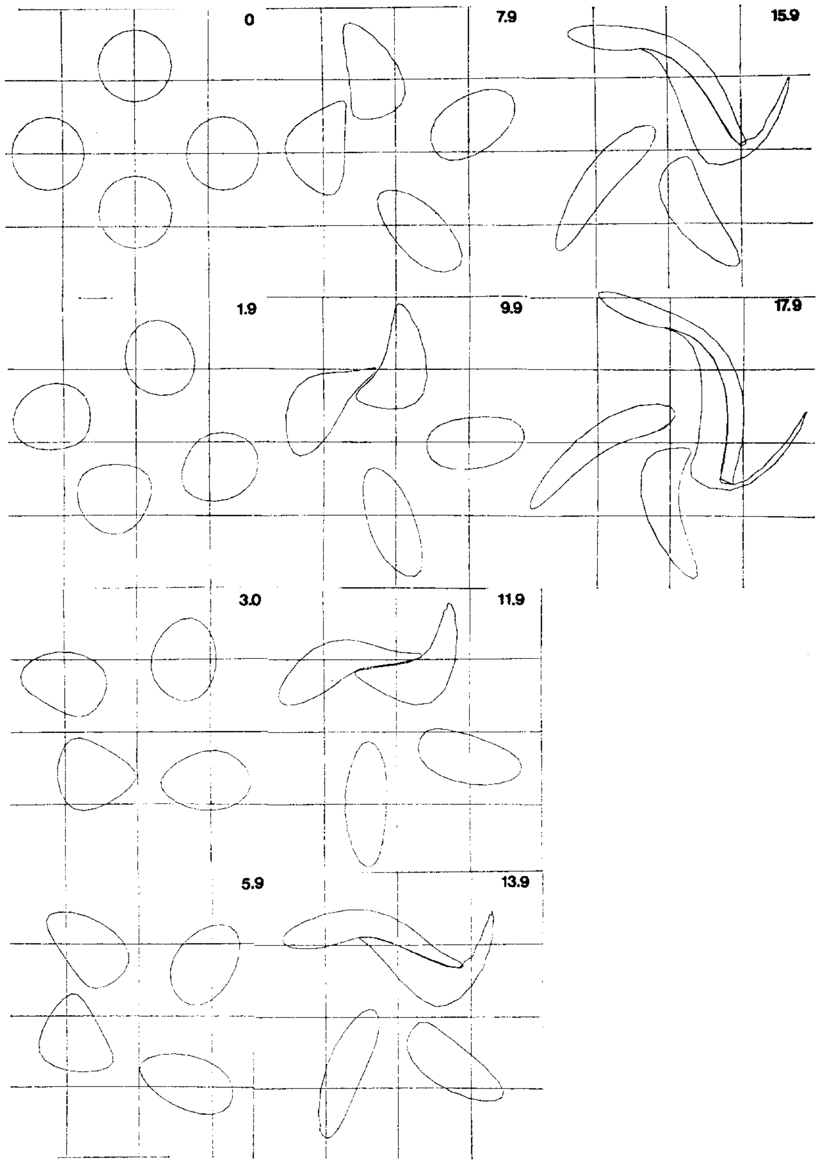

FIG. 4. Interaction of $J = 4$ initially circular FAVR's of diameter 1.0. The centers are located initially on a circle of radius 1.16 ($N = 20$ segments per circle, $\Delta t = 0.2$). $[\omega] = +1$ and the gross rotation is clockwise.

4. Discussion

We have presented a contour dynamics algorithm for incompressible inviscid fluids (Euler equations) in two dimensions. We assumed piecewise constant FAVR's and discretized the contour by assuming an N-node polygon. For moderate times we have presented high-resolution results obtained with a moderate amount of computation. To extend the algorithm to long times, one must examine errors in spatial discretization and temporal advancement. As contours elongate and merge, one must add and delete nodes as described in [5] for the Vlasov equation. Since there is no underlying mesh, this is an efficient scheme for very high Reynolds' number flows.

A variety of wave and breaking phenomena has been observed in Figs. 3 and 4. When the distance of closest approach of two or more deformable FAVR boundaries is less than their "diameter," self- and mutual interactions cause vortex "pulsation," temporary-and-permanent coalescence, and the ejection of vortex arms. Such microscale phenomena are excluded *a priori* if one deals with nondeformable vortex FAVR's or introduces *ad hoc* procedures for vortex coalescence [9].

Work is now in progress to generalize the method of contour dynamics for:

(1) stratified media described by the Boussinesq equations in two dimensions;

(2) periodic boundary conditions in x.

We will apply the latter algorithm to study the stability of the asymmetric vortex street. Von Kármán performed a linear stability analysis for a street of *point* vortices and found *marginal* stability only for a transverse-to-longitudinal separation of ($b/a = 0.281$). All other b/a ratios were unstable. Kochin *et al.* [10] showed that all point vortex streets are unstable if nonlinear terms are properly included. We will seek to validate the conjecture of Christiansen and Zabusky [4]: that an asymmetric street where $(b/a) = 0.281$ is stable because of self-consistent wavelike deformations of the FAVR boundaries.

Acknowledgments

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